Resolution Analysis in Full Waveform Inversion
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ABSTRACT: We propose a new method for the quantitative resolution analysis in full seismic waveform inversion that overcomes the limitations of classical synthetic inversions while being computationally more efficient and applicable to any misfit measure. The method rests on (1) the local quadratic approximation of the misfit functional in the vicinity of an optimal Earth model, (2) the parametrisation of the Hessian in terms of a parent function and its successive derivatives, and (3) the computation of the space-dependent parameters via Fourier transforms of the Hessian, calculated with the help of adjoint techniques. In the simplest case of a Gaussian approximation we can infer rigorously defined 3-D distributions of direction-dependent resolution lengths and the image distortion induced by the tomographic method. We illustrate these concepts with a realistic full waveform inversion for upper-mantle structure beneath Europe. While the examples presented in this paper are rather specific, the underlying idea is very general. It allows for problem-dependent variations of the theme and for adaptations to exploration scenarios and other wave equation based tomography techniques that employ, for instance, georadar or microwave data.

INTRODUCTION AND PROBLEM STATEMENT: Full waveform inversion is a tomographic technique, based on numerical wave propagation combined with adjoint methods for the computation of Fréchet kernels. The accurate and complete solution of the seismic wave equation ensures that information from the full seismogram can be used for the purpose of improved tomographic models. Originally conceived in the late 1970’s and early 1980’s, realistic applications have become feasible only recently. While the tomographic method itself has advanced substantially, an essential aspect of the inverse problem has been ignored almost completely, despite its obvious socio-economic relevance: The quantification of resolution and uncertainties.

POINT-SPREAD FUNCTIONS – AN ASTRONOMICAL DETOUR: As a preliminary, we briefly review the concept of a point-spread function, upon which the following developments are based.

Fig. 1a: Assume we observe a star with a perfect telescope. This perfect telescope images the star in the form of one localised peak in the light intensity.

Fig. 1b: With a real, imperfect telescope, we observe a smeared intensity peak, commonly referred to as the point-spread function (PSF).

Fig. 1c: For two neighbouring stars, the PSFs add up. The stars can still be distinguished because their separation is more than 2σ.

Fig. 1d: When the distance is less than 2σ, the super-position of the PSFs has only one maximum. We thus observe one single blob instead of 2 stars.

Our example motivates the definition of a resolution length as 2σ. This is a very intuitive concept, would be equally convenient in seismic tomography where we observe structural heterogeneities instead of stars.

THE TOMOGRAPHIC POINT-SPREAD FUNCTION: On Earth, we want to solve a tomographic problem by minimising a suitable misfit functional J[ψ]. In the vicinity of the optimal model parameters ψ, the tomographic equivalent of the point-spread function is given by the Hessian of J[ψ]:

\[ \mathbf{H}(\psi) \approx \frac{\partial^2 J[\psi]}{\partial \psi \partial \psi^T} \]

structural heterogeneity... seen through the tomographic telescope

To progress towards a quantitative resolution analysis, we must gain efficient access to the Hessian. The Hessian applied to any parameter vector \( [\mathbf{H}(\psi)]_{ij} \), can be computed easily via an extension of the well-known adjoint method, that requires 2 forward and 2 adjoint simulations (Fichtner & Trampert, 2011).

A REALISTIC EXAMPLE: To illustrate the PSF concept, we consider a full waveform inversion for upper-mantle structure beneath Europe:

In contrast to the astronomical toy example from Fig. 1, seismological PSFs are not generally symmetric. This means, in particular, that the resolution length is direction-dependent.

APPROXIMATION OF THE SEISMOLOGICAL PSF: ideally, we would like to compute a PSF for every point in space. Since this would be prohibitively expensive, we propose to construct an approximation of the PSF. For this, we take advantage of

1. the nearly Gaussian shape of the PSF, as seen, for instance, in fig. 3, and
2. the efficient computation of \( \mathbf{H}(\psi, \phi) \) for any \( \phi \).

The approximation of the Hessian, i.e. the PSF, proceeds as follows:

1. Parametrisation of the Hessian in terms of a Gaussian with space-dependent amplitude and covariance:

\[ \mathbf{H}(\psi, \phi) \approx \mathbf{H}(\psi, \phi) \mathbf{A}(\psi) \mathbf{A}(\psi)^T \]

This simplistic parametrisation can be generalised, using, for instance, Gram-Charlier expansions.

2. Application of a sinusoidal parameter vector to the Hessian:

\[ \mathbf{H}(\psi, \phi) = \mathbf{H}(\psi, \phi) \mathbf{A}(\psi) \mathbf{A}(\psi)^T \]

This operation yields Fourier transforms of the Hessian for specific wavenumber vectors \( k \).

3. Solution of an algebraic system for the parameters of the Gaussian approximation: The Gaussian approximation of the Hessian has a total of 7 space-dependent parameters that can be uniquely determined from the Fourier transforms for 7 independent wavenumber vectors.

APPLICATION: With the machinery outlined above, we can return to the quantification of resolution in the tomographic model in fig. 2.

The space-dependent approximation of the Hessian naturally provides estimates of the direction-dependent resolution length at every point in the model. This is shown in fig. 4.

In addition to quantifying the distribution of resolution lengths, we can compute the distortion of the real Earth structure introduced by the tomography.

A point-perturbation in the Earth is always mapped into a blurred version of itself, and the centre of mass of the blurred image is generally at a different position.

Thus, what we see in the tomographic model may actually be somewhere else in the Earth. This distortion, i.e. the distance between reality and image is shown in fig. 5.

Fig. 5: Distortion at 300 km depth. The length of the red line on top corresponds to 300 km. Throughout most of central Europe the distortion is small, but there are regions where heterogeneities appear in the wrong places.

CONCLUSIONS: So far, we can draw the following conclusions from this work:

1. We developed the first method for quantitative resolution analysis in full seismic waveform inversion.
2. The method rests on a parametrisation of the Hessian, and the computation of the space-dependent parameters with the help of Fourier transforms that can be computed efficiently with a slight extension of the adjoint method.
3. From the approximated Hessian, we can derive several indicators of resolution, including direction-dependent resolution lengths and distortions.
4. The computational costs of our method correspond to the costs of a full waveform inversion with only 6 iterations. (Typically, more than 20 iterations are needed to achieve satisfactory results.) Our method is therefore more efficient than synthetic inversions, while providing a much more complete picture of resolution.

COURRILLS AND OUTLOOK: The approximation of the Hessian can furthermore be used as a pre-conditioner in gradient-type optimisation schemes, for adaptive parametrisation independent of ray theory, and for objective functional design that aims at maximising resolution. While our examples presented are rather specific, the underlying idea is very general. It allows for problem-dependent variations of the theme and for adaptations to exploration scenarios and other wave equation based tomography techniques that employ, for instance, georadar or microwave data.