Resolution Analysis in Full Waveform Inversion

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ABSTRACT: We propose a new method for the quantitative resolution analysis in full seismic waveform inversion that overcomes the limitations of classical synthetic inversions while being computationally more efficient and applicable to any misfit measure. The method rests on (1) the local quadratic approximation of the misfit functional in the vicinity of an optimal Earth model, (2) the parametrisation of the Hessian in terms of a parent function and its successive derivatives, and (3) the computation of the space-dependent parameters via Fourier transforms of the Hessian, calculated with the help of adjoint techniques. In the simplest case of a Gaussian approximation we can infer rigorously defined 3-D distributions of direction-dependent resolution lengths and the image distortion introduced by the tomographic method. We illustrate these concepts with a realistic full waveform inversion for upper-mantle structure beneath Europe. While the examples presented in this paper are rather specific, the underlying idea is very general. It allows for problem-dependent variations of the theme and for adaptations to exploration scenarios and other wave equation based tomography techniques that employ, for instance, georadar or microwave data.

INTRODUCTION AND PROBLEM STATEMENT: **Full waveform inversion** is a tomographic technique, based on **numerical wave propagation** combined with **adjoint methods** for the computation of Fréchet kernels. The accurate and complete solution of the seismic wave equation ensures that information from the full seismogram can be used for the purpose of improved tomographic models. Originally conceived in the late 1970's and early 1980's, realistic applications have become feasible only recently. While **the tomographic method itself has advanced substantially**, an essential aspect of the inverse problem has been ignored almost completely, despite its obvious socio-economic relevance: **The quantification of resolution and uncertainties**.

POINT-SPREAD FUNCTIONS – AN ASTRONOMICAL DETOUR: As a preliminary, we briefly revise the concept of a point-spread function, upon which the following developments are based.

Fig. 1a: Assume we observe a star with a **perfect telescope**. This perfect telescope images the star in the form of one **localised peak** in the light intensity.

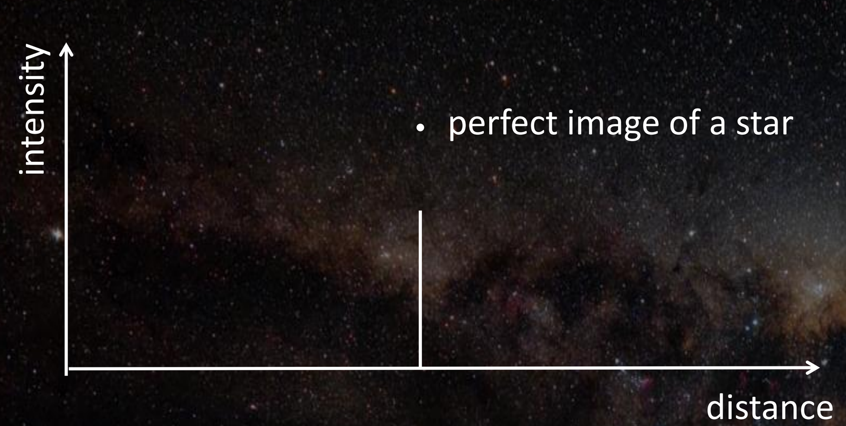


Fig. 1b: With a **real, imperfect telescope**, we observe a smeared intensity peak, commonly referred to as the **point-spread function (PSF)**.

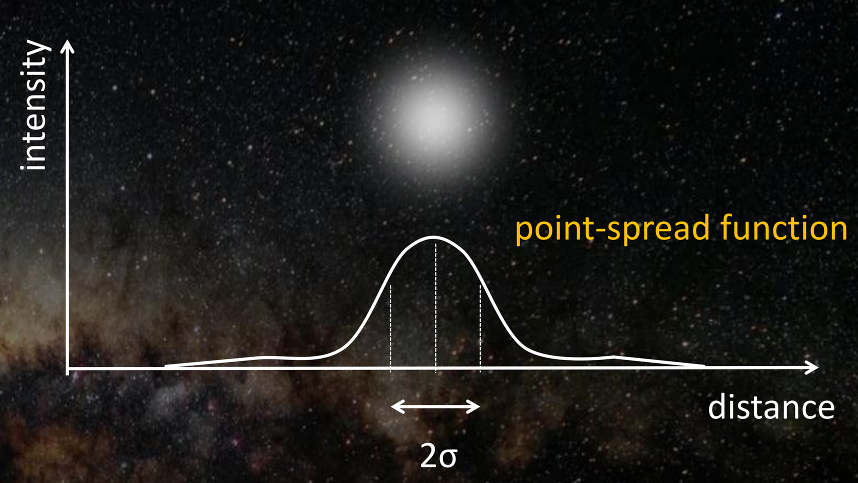


Fig. 1c: For two neighbouring stars, the PSF's add up. The stars can still be distinguished because their separation is more than 2σ.

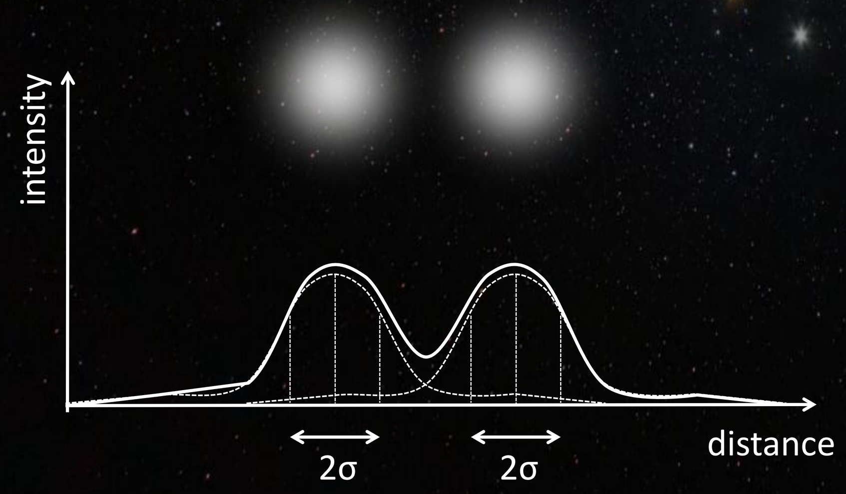
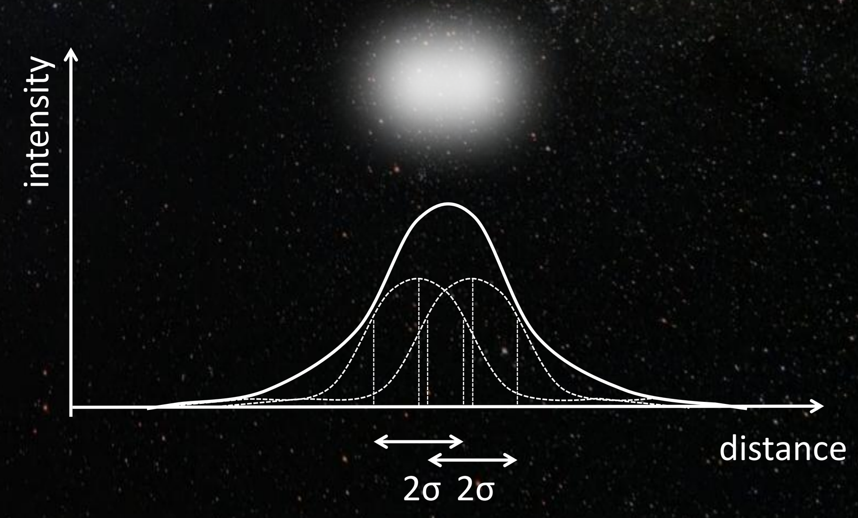


Fig. 1d: When the distance is less than 2σ, the super-position of the PSF's has only one maximum. We thus observe one single blob instead of 2 stars.



Our example motivates the **definition of a resolution length as 2σ**. This is a very intuitive concept, would be equally convenient in seismic tomography where we observe structural heterogeneities instead of stars.

THE TOMOGRAPHIC POINT-SPREAD FUNCTION: On Earth, we want to solve a tomographic problem by minimising a suitable misfit functional $\chi(\mathbf{p})$.

In the vicinity of the optimal model parameters $\mathbf{p}_{opt}$, the tomographic equivalent of the point-spread function is given by the Hessian of χ:

PSF = Hessian of χ at $\mathbf{p}_{opt}$: $\mathbf{H}(\mathbf{x},\mathbf{x}_0)$

To progress towards a quantitative resolution analysis, we must gain efficient access to the Hessian.

The Hessian applied to any parameter vector, $\int \mathbf{H}(\mathbf{x},\mathbf{x}')\mathbf{p}(\mathbf{x}')\,d\mathbf{x}'$, can be computed easily via an extension of the well-known adjoint method, that requires 2 forward and 2 adjoint simulations (*Fichtner & Trampert*, *2011*).

A REALISTIC EXAMPLE: To illustrate the PSF concept, we consider a full waveform inversion for upper-mantle structure beneath Europe:

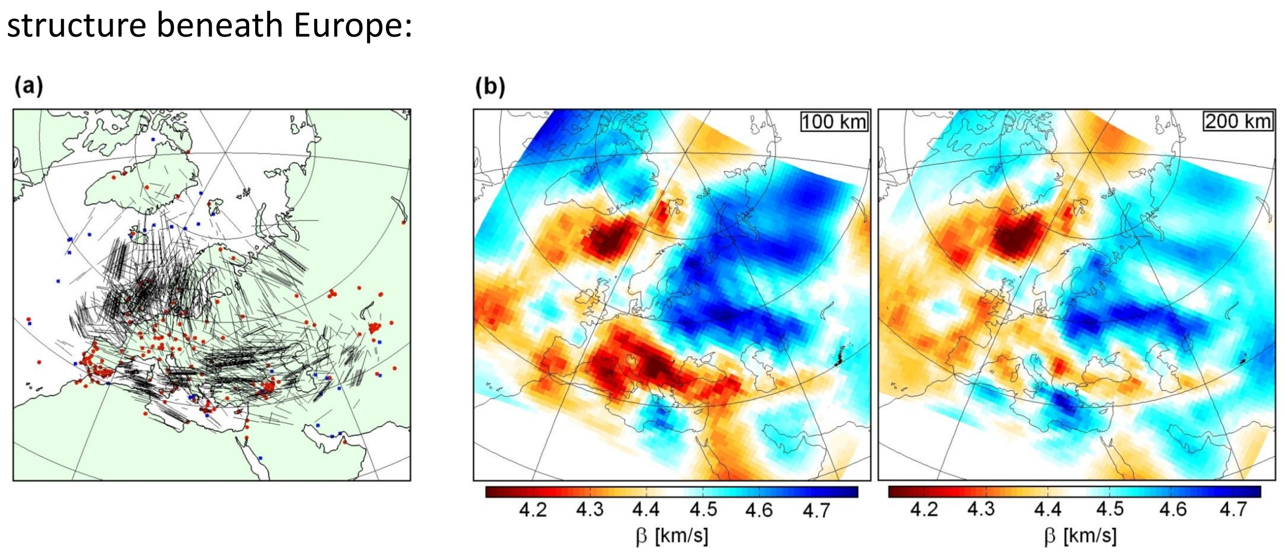


Fig. 2: (a) Earthquake hypocentres (blue dots), seismic stations (red dots) and ray path segments for the data set used in the inversion. The dominant period of the data is 100 s. **(b)** S velocity distribution at 100 km and 200 km depth.

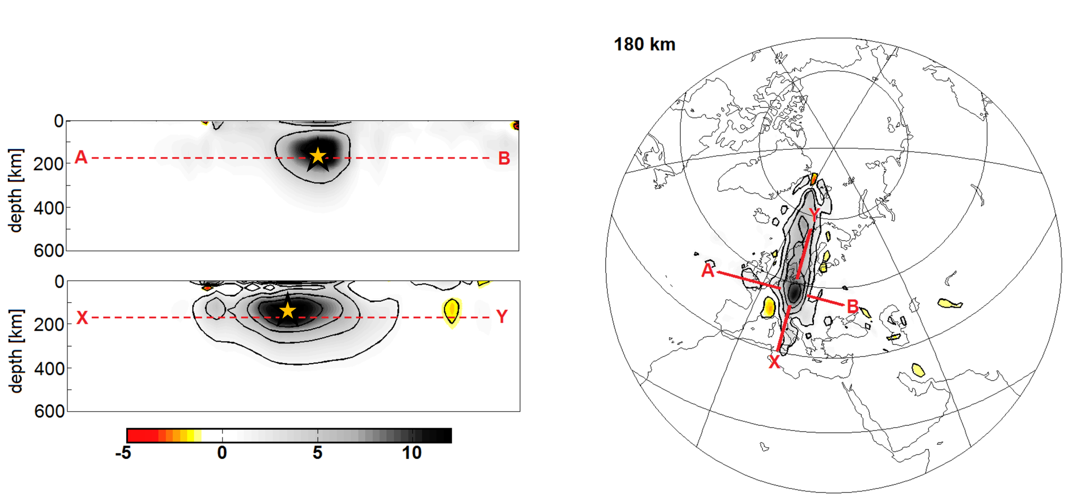


Fig. 3: Seismic point-spread function for a point-localised S velocity heterogeneity at the position of the yellow star, denoted by $\mathbf{x}_0$.

$$\mathrm{PSF}(\mathbf{x}) = \mathbf{H}(\mathbf{x},\mathbf{x}_0) = \int \mathbf{H}(\mathbf{x},\mathbf{x}')\,\delta(\mathbf{x}_0-\mathbf{x}')\,d\mathbf{x}'$$

Mathematically, the PSF is a continuous version of a column of the Hessian.

Physically, the PSF defines a volume, where different heterogeneities cannot be distinguished.

In contrast to the astronomical toy example from fig. 1, seismological PSF's are not generally symmetric. This means, in particular, that the resolution length is direction-dependent.

APPROXIMATION OF THE SEISMOLOGICAL PSF: Ideally, we would like to compute a PSF for every point in space. Since this would be prohobitively expensive, we propose to construct an approximation of the PSF. For this, we take advantage of

1. the nearly Gaussian shape of the PSF, as seen, for instance, in fig. 3, and

2. the efficient computation of $\int \mathbf{H}(\mathbf{x},\mathbf{x}')\mathbf{p}(\mathbf{x}')\,d\mathbf{x}'$ for any $\mathbf{p}(\mathbf{x}')$.

The approximation of the Hessian, i.e. the PSF, proceeds as follows:

1. **Parametrisation of the Hessian** in terms of a Gassian with space-dependent amplitude and covariance:

$$H(x,y) \approx A(x)\, e^{-(x-y)^T C(x)(x-y)}$$

A(x): position-dependent **amplitude**
C(x): position-dependent **width**

This simplistic parametrisation can be generalised, using, for instance, Gram-Charlier expansions.

2. **Application of a sinusoidal parameter vector** to the Hessian:

$$\tilde{H}(x,k) = \int H(x,y)\, e^{-ik^T y}\, dy \;\propto\; A(x)\, e^{i k^T C^{-1}(x) k}$$

This operation yields Fourier transforms of the Hessian for specific wavenumber vectors **k**.

3. **Solution of an algebraic system for the parameters of the Gaussian approximation**: The Gaussian approximation of the Hessian has a total of 7 space-dependent parameters that can be uniquely determined from the Fourier transforms for 7 independent wavenumber vectors.

APPLICATION: With the machinery outlined above, we can return to the quantification of resolution in the tomographic model in fig. 2.

The space-dependent approximation of the Hessian naturally provides estimates of the **direction-dependent resolution length at every point in the model**. This is shown in fig. 4 .

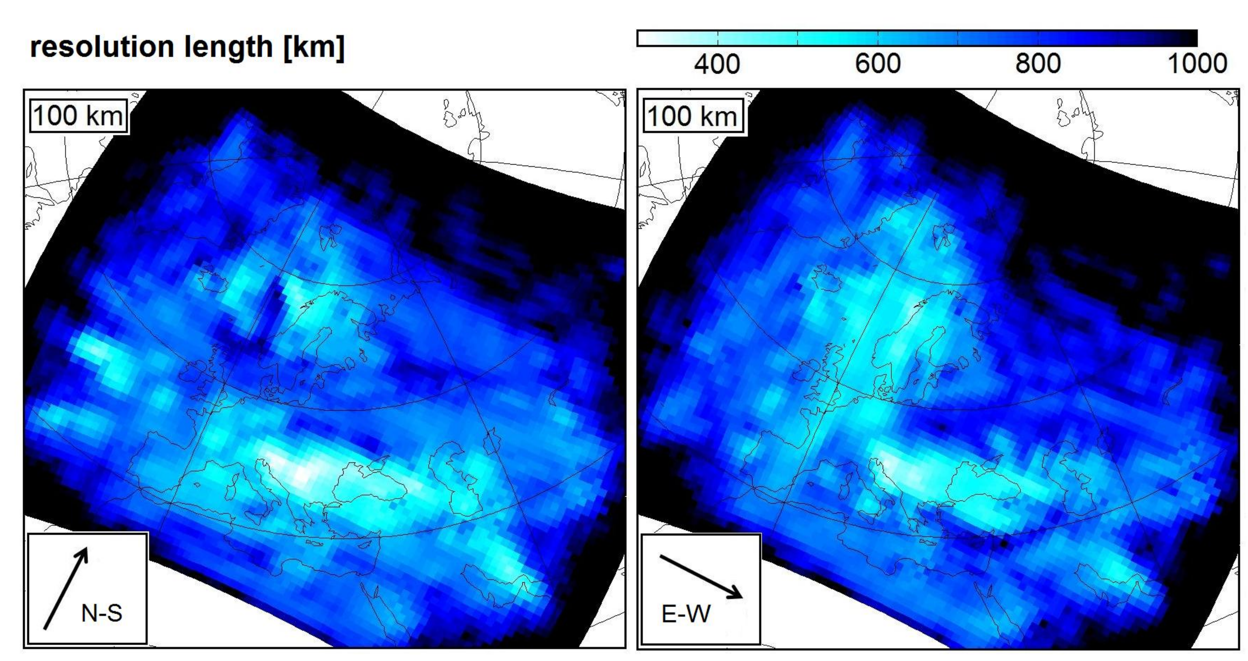
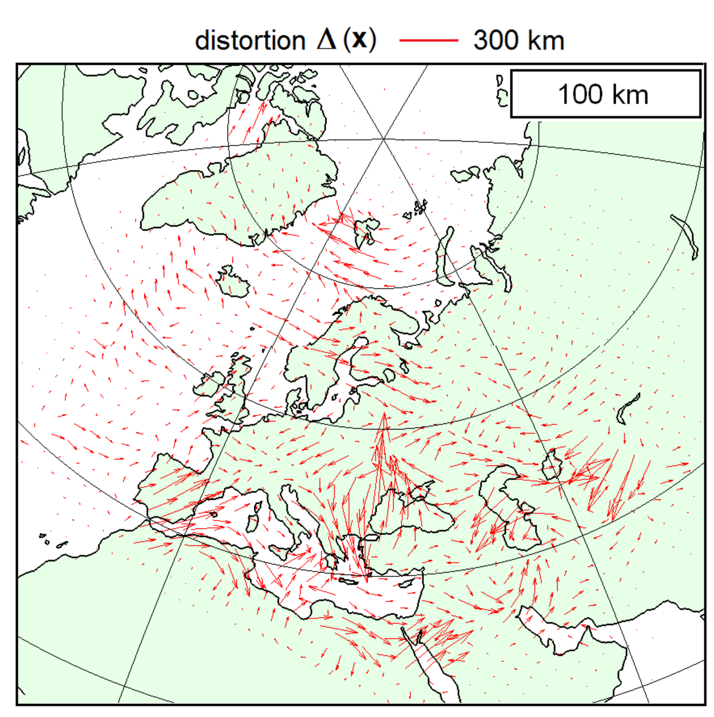


Fig. 4: Direction-dependent resolution length at 100 km depth for the tomographic model shown in fig. 3. **Left:** The resolution length in N-S direction indicates the N-S extension of the PSF. It reaches a minimum of around 300 km beneath the Balkans. Structure in the dark blue and black regions is practically unresolved. **Right:** The resolution in E-W direction is similar to the one in N-S direction, but an additional minimum beneath the North Atlantic appears. This results from the many N-S oriented ray paths from earthquakes along the North Atlantic ridge to stations in central Europe. In general, resolution in one direction is determined by the amount of information (seismic waves) propagating in the perpendicular direction.



In addition to quantifying the distribution of resolution lengths, we can compute the distortion of the real Earth structure introduced by the tomography.

A **point-perturbation** in the Earth is always mapped into a **blurred version of itself**, and the centre of mass of the blurred image is generally at a different position.

Thus, **what we see in the tomographic model may actually be somewhere else in the Earth**. This distortion, i.e. the distance between reality and image is shown in fig. 5.

Fig. 5: Distortion at 100 km depth. The length of the red line on top corresponds to 300 km. Throughout most of central Europe the distortion is small, but there are regions where heterogeneities appear in the wrong places.

CONCLUSIONS: So far, we can draw the following conclusions from this work:

1. We developed the **first method for quantitative resolution analysis in full seismic waveform inversion**.

2. The method rests on a **parametrisation of the Hessian**, and the computation of the space-dependent parameters with the help of **Fourier transforms** that can be computed efficiently with a slight **extension of the adjoint method**.

3. From the approximated Hessian, we can derive several indicators of resolution, including **direction-dependent resolution lengths and distortions**.

4. The computational costs of our method correspond to the costs of a full waveform inversion with only 6 iterations. (Typically, more than 20 iterations are needed to achieve satisfactory results.) **Our method is therefore more efficient than synthetic inversions, while providing a much more complete picture of resolution**.

COROLLARIES AND OUTLOOK: The approximation of the Hessian can furthermore be used as a **pre-conditioner** in gradient-type optimisation schemes, for **adaptive parametrisation** independent of ray theory, and for **objective functional design** that aims at maximising resolution.
While our examples presented are rather specific, the underlying idea is very general. It allows for problem-dependent variations of the theme and for adaptations to exploration scenarios and other wave equation based tomography techniques that employ, for instance, georadar or microwave data.