Why you should or should not use stabilised Q_1Q_1 elements for geodynamical modelling

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Abstract

Despite well-documented drawbacks, the bi-tri-linear velocity - constant pressure (Q1P0) element is still the workhorse of many Finite Element geodynamics codes. With the advent of modern parallel iterative solvers and ever more powerful high performance computers, a new generation of codes now relies on quadratic elements, which are stable and yield better accuracy. However, two exceptions come to mind in the past decade: the Rhea code [1,5] and the GALE code [2] which both rely (or relied, in the second case) on the stabilised low-order element Q1Q1 element. This element is linear for both velocity and pressure. While Rhea was very successful for large-scale adaptive mantle convection simulation (Stadler et al, Science 2010), the GALE manual states: "[Like the Q1P0 element,] this formulation has its own instability that is fixed by adding an artificial compressibility. In principle, this artificial compressibility should be small and get smaller as resolution increases. In practice, for realistic geologic problems, the artificial compressibility was far too large and dramatically altered the dynamics."

1. Conservation equations

The mechanical behavior of Earth materials is described by means of the Stokes equations:

$$\nabla \cdot \left(\mu(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T)\right) - \nabla p + \rho \boldsymbol{g} = \boldsymbol{0} \quad (1)$$
$$\nabla \cdot \boldsymbol{v} = 0 \quad (2)$$

Eq. (1) is the momentum conservation equation and Eq. (2) is the mass conservation equation for incompressible fluids.

2. Discretisation

Although stabilised bi-tri-linear velocity and pressure (Q_1Q_1) element were previously proposed [7,9] the one proposed in [4,6] has a clear distinct advantage: it is simple to implement and does not involve any tunable parameter. Through careful analysis it was showed that the element is stable and that the pressure modes do not occur.

The FE discretisation of the Stokes equations yields the following linear system:

$$\begin{pmatrix} \mathbb{K} & \mathbb{G} \\ \mathbb{G}^T & -\mathbb{C} \end{pmatrix} \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{P} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{h} \end{pmatrix}$$

where

$$\mathbb{C}(p,q) = \frac{1}{\mu}\int_\Omega (p - \pi(p))(q - \pi(q)) d\Omega$$

and where π actuallys stands for a projection operator simply given by:

$$\pi(p) = \frac{1}{V}\int_\Omega p d\Omega = \bar{p}$$

The system is solved by means of a Preconditioned Conjugate Gradient algorithm applied to the Schur complement.

References

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[4] Dohrmann and Bochev, IJNMF 46, 2004.
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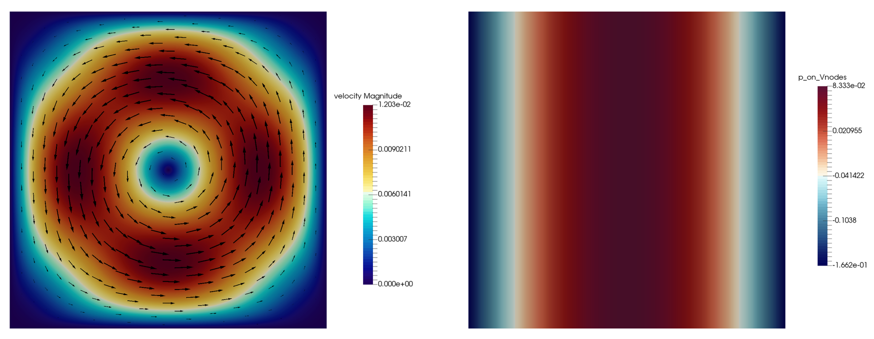
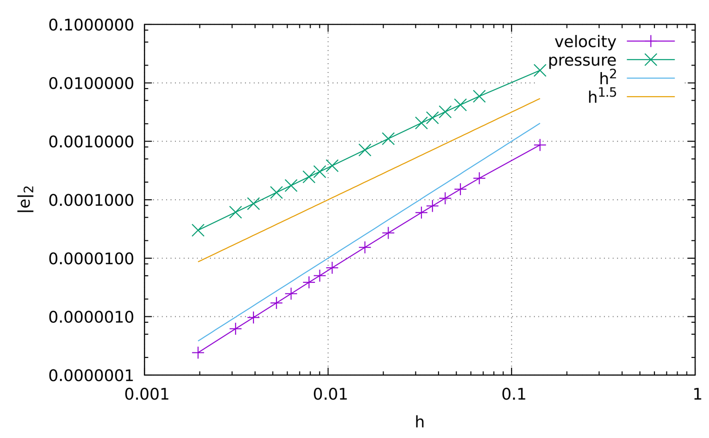
3. Simple analytical problem

The domain is square $\Omega = [0,1] \times [0,1]$, and boundary conditions are such that $\boldsymbol{v} = \boldsymbol{0}$ on Γ. The viscosity is set to μ=1. The exact solution is then given by [3]:

$$u(x,y) = 2x^2(1-x)^2(y - 3y^2 + 2y^3) \quad (3)$$
$$v(x,y) = -2y^2(1-y)^2(x - 3x^2 + 2x^3) \quad (4)$$
$$p(x,y) = x(1-x) - 1/6 \quad (5)$$

The body force vector is obtained by inserting the exact solution in Eq. (1). Note that $\int_\Omega p dV = 0$ by construction.





$\Rightarrow$ Q_1Q_1 element pair well implemented.

4. SolCx & SolKz

These benchmarks are intended to test the accuracy of the solution to a problem that has a large jump in the viscosity along a line through the domain. Such situations are common in geophysics: for example, the viscosity in a cold, subducting slab is much larger than in the surrounding, relatively hot mantle material.

The SolCx benchmark computes the Stokes flow field of a fluid driven by spatial density variations, subject to a spatially variable viscosity. Specifically, the domain is $\Omega = [0,1]^2$, gravity is $\boldsymbol{g} = (0,-1)^T$ and the density is given by

$$\rho(x,y) = \sin(\pi y)\cos(\pi x) \quad (6)$$

Boundary conditions are free slip on all of the sides of the domain and the temperature plays no role in this benchmark. The viscosity is prescribed as follows:

$$\mu(x,y) = \begin{cases} 1 & for\ x < 0.5 \\ 10^6 & for\ x > 0.5 \end{cases} \quad (7)$$

The SolCx benchmark was previously used in [10] (references to earlier uses of the benchmark are available there) and its analytic solution is given in [11]. It has been carried out in [12,13]. Note that the source code which evaluates the velocity and pressure fields for both SolCx and SolKz is distributed as part of the open source package Underworld [14], http://underworldproject.org).

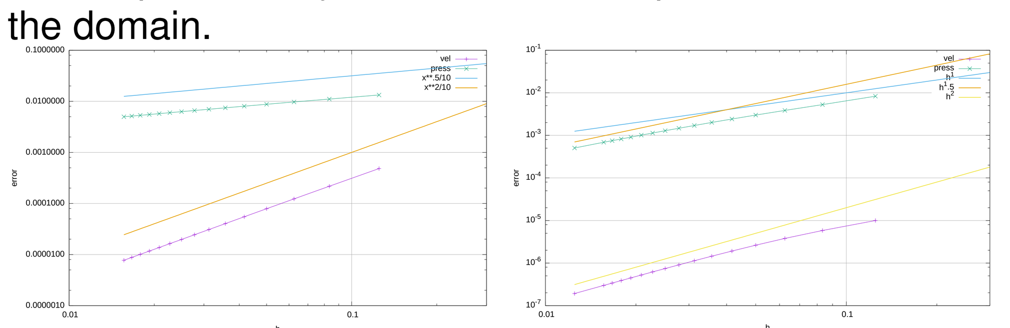
The SolKz benchmark [15] is similar to the SolCx benchmark but the viscosity is now a function of the space coordinates:

$$\mu(y) = \exp(2By) \quad \text{with} \quad B = 13.8155 \quad (8)$$

The forcing is again chosen by imposing a spatially variable density variation as follows:

$$\rho(x,y) = \sin(2y)\cos(3\pi y) \quad (9)$$

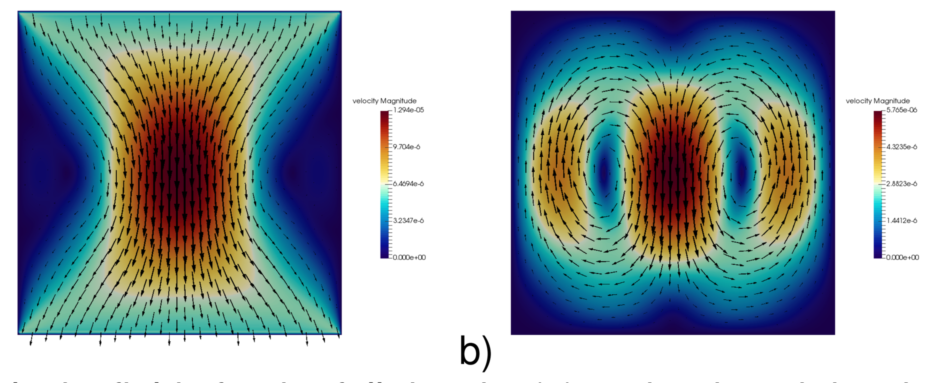
Free slip boundary conditions are imposed on all sides of the domain.



$\Rightarrow$ Stokes solver can handle large viscosity contrasts

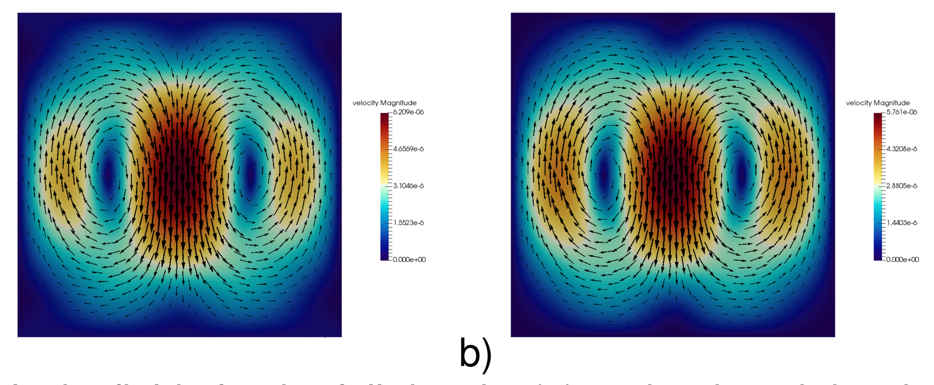
5. Stokes sphere

The domain is a unit square with no-slip boundary conditions. Gravity is $\boldsymbol{g} = (0,-1)$. The fluid has a density $\rho_0 = 1$ and the sphere has a density $\rho_s = \rho_0 + \delta\rho$ with $\delta\rho \in [10^{-3} : 1]$. There is no viscosity contrast between the sphere and the fluid. Simulations are also run with reduced densities, i.e. $\rho_0 = 0$ and $\rho_s = \delta\rho$.



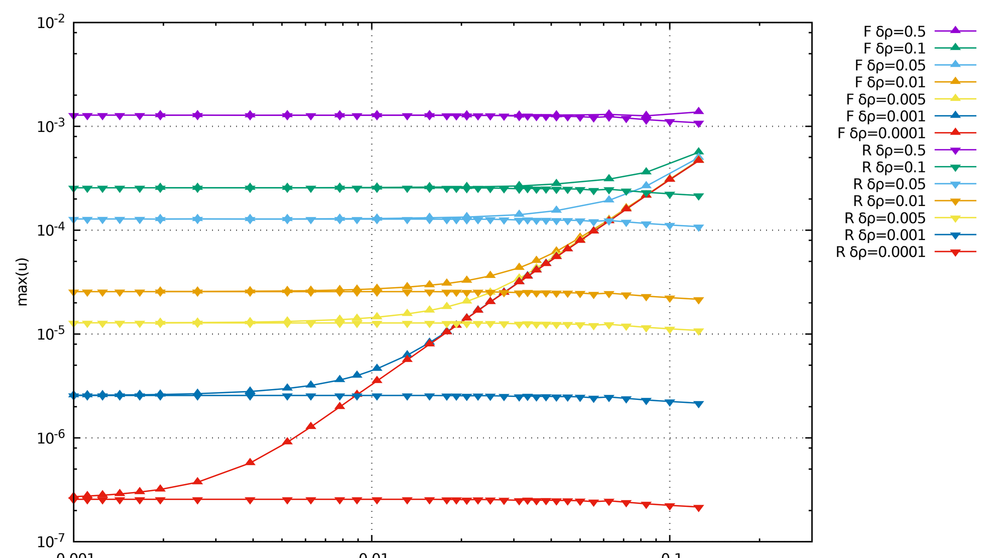
a) b)

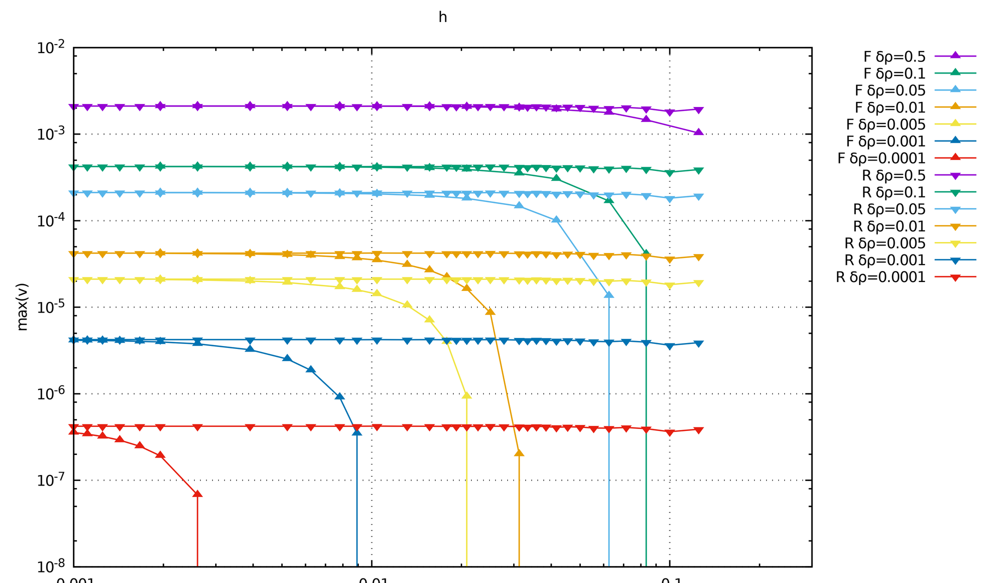
Velocity fields for the full density (a) and reduced density (b) with $\delta\rho = 0.001$ and resolution 128x128.



a) b)

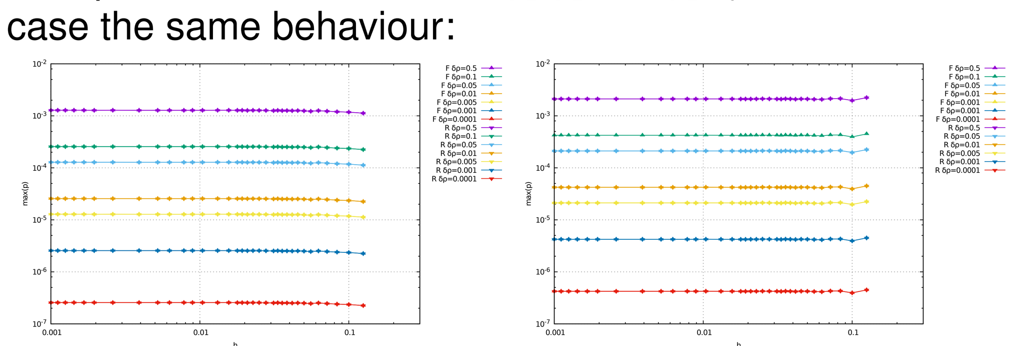
Velocity fields for the full density (a) and reduced density (b) with $\delta\rho = 0.001$ and resolution 512x512.





$\Rightarrow$ When $\delta\rho = 10^{-4}$ a resolution of 1024^2 is not enough to arrive at the expected physical velocity field!

This problem in inherent to Q_1Q_1 as Q_1P_0 does not showcase the same behaviour:



6. Conclusion

- Q1Q1 element relatively easy to implement and easy to incorporate in a Schur complement approach solving procedure.
- When using full density a *very high* resolution is needed to achieve mesh independent measurements for small density variations (only 10^{-3} in this case)
  $\Rightarrow$ buoyancy driven flows nearly impossible!
- When using reduced densities, this problem is alleviated but recovered pressure is overpressure, not total pressure.
  $\Rightarrow$ Free surface flows impossible!

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