Langevin Regression for a softglass system

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Introduction

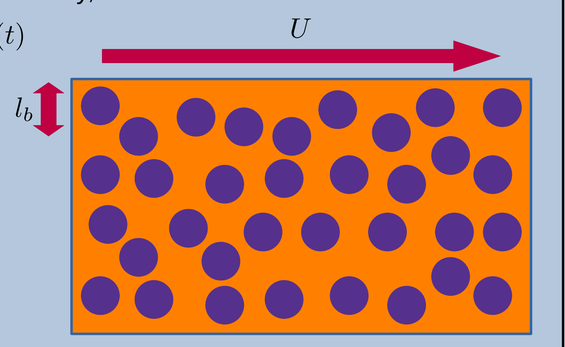
- A softglass is an emulsion, e.g. of oil and water
- It can display a range of behaviours when different kinds and strengths of forcing are applied
  - Viscoelastic wave propagation
  - Flow
  - Stick-slip
  - Shear-bending (localised flow)

We can model the bulk properties of fluidity (inverse viscosity) and stress of the softglass with [1]: a stochastic differential equation for ϕ, the square root of fluidity,

$$\dot{\phi}(t) = -R\phi + m(\sigma)\phi^3 - |\phi|\,\phi^3 + \sqrt{\epsilon_0 + \epsilon_1\phi^2}\eta(t)$$

and a Maxwell relation for the stress σ,

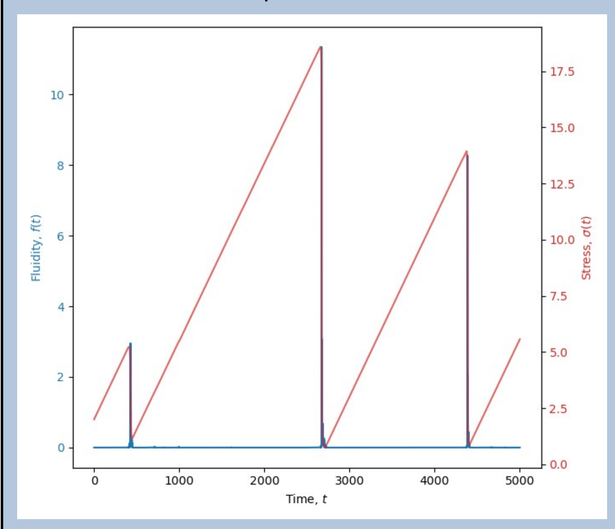
$$\dot{\sigma}(t) = \frac{1}{\tau}\left(U - (1 - l_b)\phi^2\sigma - l_b m(\sigma)\sigma\right)$$



- Physically these equations model a Couette flow:
  - Purple droplets: Fluid 1, e.g. oil
  - Orange medium: Fluid 2, e.g. water
  - Shear-strain rate U imposed by moving the top wall
  - A flowing region of size l_b at by the moving wall
  - Periodic walls on the left and right

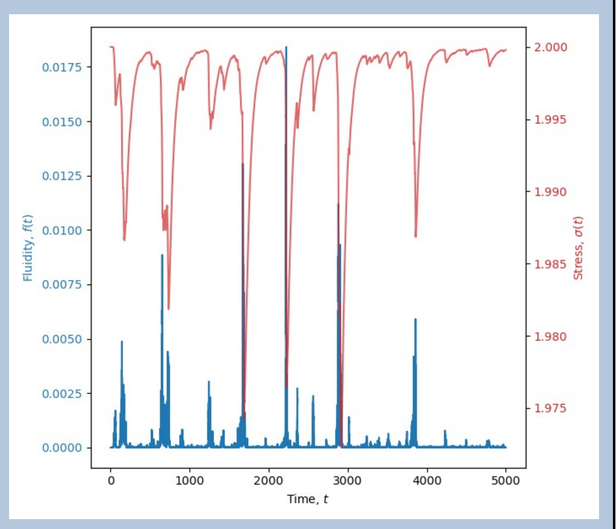
We can evolve these equations for the stick-slip and shear-bend scenarios which are controlled by the l_b parameter.

Stick-Slip scenario: $l_b = 0$



- Stress can build up infinitely
- An avalanche is a sharp delta-function like event
- Most of the stress in the system is released

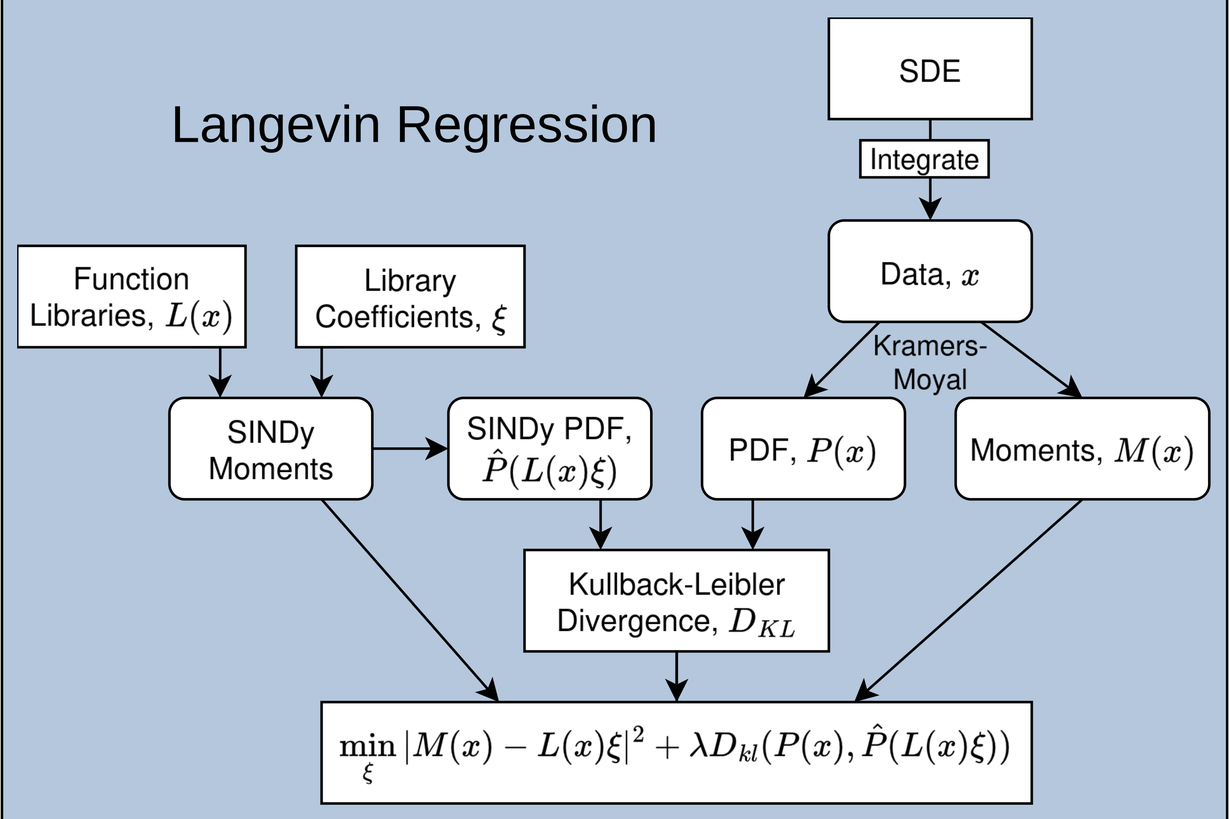
Shear-bend scenario: $l_b > 0$



- Maximum stress due to flowing region
- Avalanches last for longer
- Stress drops are small compared to the maximum

Q: Can the PDE be recovered given $\phi(t)$ and $\sigma(t)$?

Method



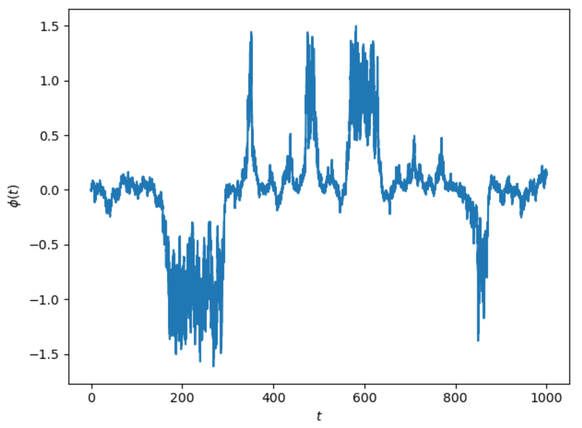
- SINDy (Sparse Identification of Non-linear Dynamics) [3]
  - (Non-linear) basis functions in a library, $L(x)$
  - Regression to find library coefficients ξ that fit the data
  - Sparsity is enforced by removing terms from the library
- Stochastic SINDy [4]
  - First moment of the PDF corresponds to the drift (deterministic) part of the SDE
  - Second moment of the PDF corresponds to the diffusion (stochastic) part of the SDE
- Langevin Regression [2]
  - Solve the Steady state Fokker-Planck equation for the PDF corresponding to the SINDy moments
  - Regularise the ill-posed problem with the Kullback-Leibler divergence of the data PDF and SINDy PDF
  - Can also find white noise approximations to coloured noise systems and systems with unresolved dynamics

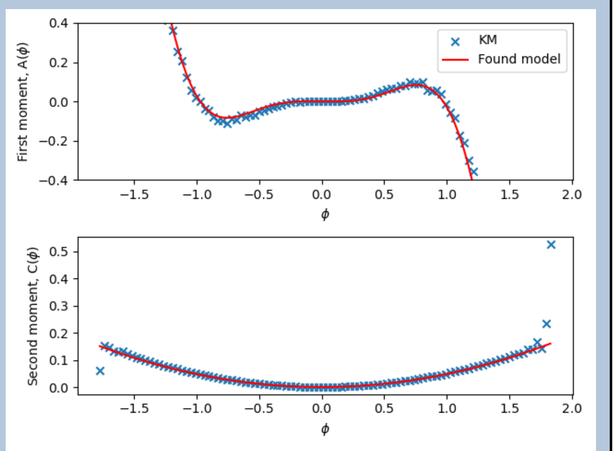
References

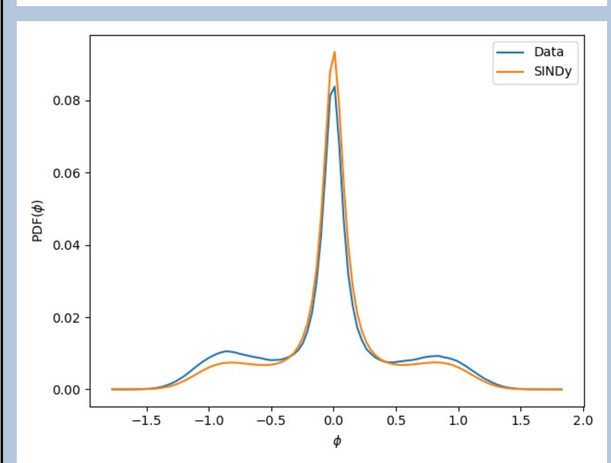
[1] Benzi et al. 2016 Cooperativity flows and shear-bandings: a statistical field theory approach. *Soft Matter*, **12**, 514
[2] Callaham et al. 2021 Nonlinear stochastic modelling with Langevin Regression. *Proc. R. Soc. A* **477**: 202110092
[3] Brunton et al. 2016 Discovering governing equations from data by sparse identification of nonlinear dynamical systems. *PNAS* **111** 15
[4] Boninsegna et al. 2018 Sparse Learning of stochastic dynamical equations. *J. Chem. Phys.* **148**, 241723

Example 1: 0D fluidity model with constant stress

- Physically corresponds to controlling the stress instead of the shear-strain rate
- Results in a pitchforking behaviour where the system switches between stable states at 0 and ± 1
- In this case the diffusion term has values $\varepsilon_0 = 10^{-3}$ and $\varepsilon_1 = 0.1$



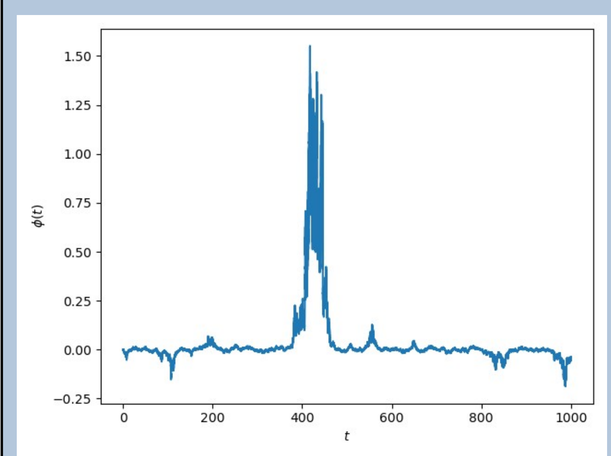


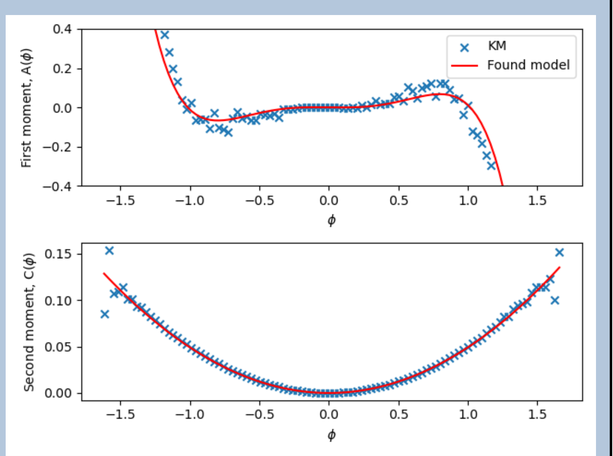


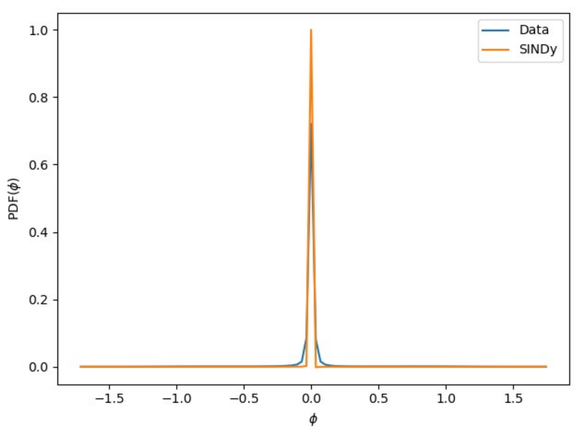
- Found drift visually fits the data very well, but has higher order terms not present in the true model
- There are terms from the true drift which are missing in the found model
- And terms in the found model which are not in the true model
- The found diffusion for the model is exactly correct
- This regression was performed with the KL regularisation turned off, and even so the PDF is remarkably well fitting

Example 2: Example 1 with lower value of ε_0

- 2 orders of magnitude lower mechanical noise
  - Smaller variation around 0
  - Excited states are visited less
- More data is thus required for the KM moments to converge
  - Especially seen at large ϕ



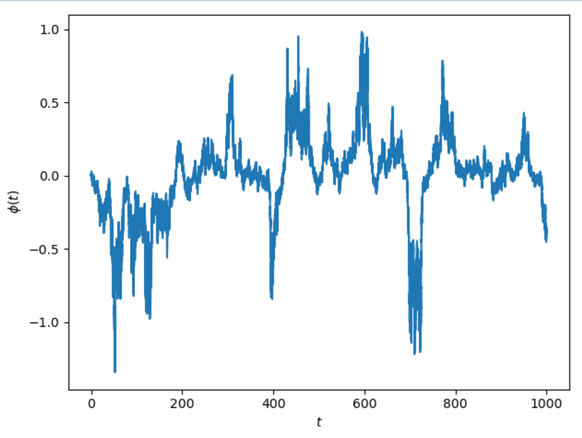


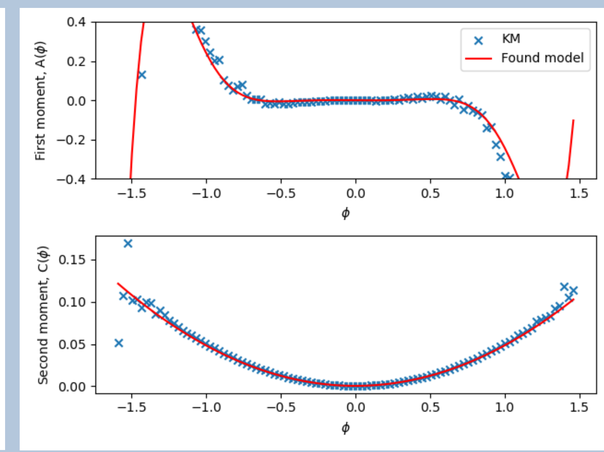


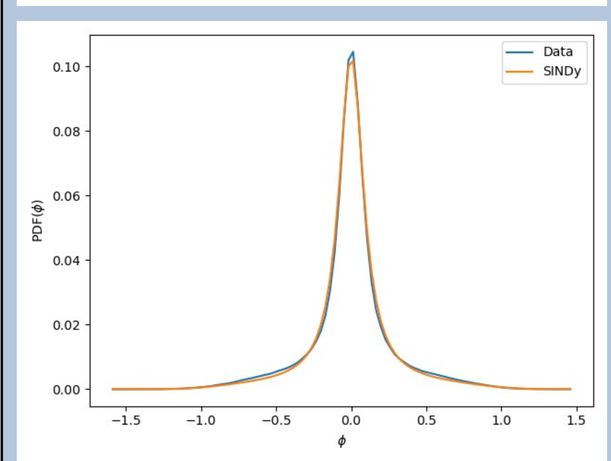
- Drift diverges from data particularly around the wells which are less frequently visited
- Found diffusion appears to correctly follow data, but is missing the very small ep0 value
- The PDF is much narrower and focused more on $\phi = 0$, though the wells at $\phi = 1$ are occasionally visited so the moment is still well defined at those values

Example 3: Example 1 with lower value of $m(\sigma)$

- Smaller $\sigma \sim 2.6 \rightarrow \sim 1.98$
  - Closer to the stress seen in the shear bending system
  - Stability of the pitchfork decreases







- The wells at $\phi = \pm 1$ have moved inwards and decreased in size
- Results in correct drift terms being more difficult to learn
- Found diffusion perfectly matches the true diffusion
- The resulting PDF is difficult to distinguish from a Gaussian